Observer design for a class of nonlinear systems with nonscalar-input nonlinear consequents

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***Abstract*— This paper presents a discrete-time Takagi-Sugeno fuzzy observer design approach for a class of nonlinear systems. Instead of including all the nonlinear terms in the membership functions, some of them are kept as nonlinear consequents, and they need to fulfill a global Lipschitz condition. The form considered permits nonlinear consequents that depend on nonscalar inputs. The design conditions are defined in terms of linear matrix inequalities, and they are less restrictive than previous conditions from the literature. Two numerical examples highlight the advantages obtained.**

I. INTRODUCTION

For systems where direct measurements of relevant state variables are not physically possible or the sensors are too expensive, estimation is used. This research field has started with the seminal works [1] and [2], and has been explored in depth for both linear and nonlinear systems.

An influential nonlinear observer design approach is presented in [3], where a slope-bound condition is used to handle the nonlinearities, assuming that each nonlinearity is a function of a given linear combination of the states. A generalization of this approach is given in [4], where less conservative design conditions are developed. Furthermore, in [5], the restriction that the nonlinearities must be a function of the same linear combination of the states is removed. They define a more general form of the nonlinearities that may depend on nonlinear combinations of the states, for example, nonlinearities like $\cos(x_1 x_2)$. Several extensions of this work can be found, see e.g. [6], [7], [8], [9], [10].

In particular, the discrete-time observer design problem when the dynamics of the system contain such nonlinearities is presented in [10], which is the starting point of our paper. From the structural point of view, we consider a similar model to the one in [10]. There, the models used are Linear Parameter Varying (LPV), which are in direct analogy to the Takagi-Sugeno (TS) fuzzy models that we consider.

Specifically, we present in this paper nonlinear observer design approaches for a class of nonlinear systems represented as TS models with nonscalar-input nonlinear consequents. We provide the following improvements compared to the results in [10].

First, we decouple the direct dependency of the observer gains on the Lyapunov function by adding an extra degree of freedom with a non-symmetric fuzzy matrix, Q_z. Second, to further relax the design conditions, we consider a fuzzy Lyapunov function instead of a quadratic one. Finally, in [10] the design conditions in the main theorem are presented as bilinear matrix inequalities (BMIs). There, two options are presented to obtain sufficient LMI conditions. In this paper, next to the two options of [10], we propose an alternative approach to obtain LMIs. The design conditions in this approach are in certain cases less restrictive than in the other two approaches.

On the other hand, there are also related works in the TS fuzzy literature, see e.g. [11], [12], [13], [14], [15]. For example, in [11] the following Lipschitz condition is used for observer design: $\|\phi_i(x(k)) - \phi_i(\hat{x}(k))\| \leq \theta_i \|R_i(x(k) - \hat{x}(k))\|$. The above inequality uses the norm of the nonlinearity, and an upper bound is defined for this norm, while the present results are based on the mean-value theorem and are less restrictive. Moreover, our approach handles each state dependency separately for every nonlinearity, and for each such state dependency a different upper bound is defined, which adds flexibility to the design conditions. In [13] the nonlinear consequents can handle only scalar inputs, while the present work can tackle multiple inputs in the nonlinear consequents.

In the sequel, following some notations, the TS fuzzy system with nonlinear consequents and the estimation problem are introduced in Section II. Section III presents the main theoretical results. To highlight the novelty of the paper two numerical examples are provided in Section IV. Finally, conclusions and future directions are presented in Section V.

Notations. Let $F = F^T \in \mathbb{R}^{n \times n}$ be a real symmetric matrix; $F > 0$ and $F < 0$ mean that F is positive definite and negative definite, respectively. I denotes the identity matrix and 0 the zero matrix of appropriate dimensions. The symbol $*$ in a matrix indicates a transposed quantity in the symmetric position, for instance $\begin{pmatrix} P & * \\ A & P \end{pmatrix} = \begin{pmatrix} P & A^T \\ A & P \end{pmatrix}$, $A + * = A + A^T$, and $AP* = APA^T$. The notation $\mathrm{bdiag}(f_1, ..., f_m)$, where $f_i \in \mathbb{R}^{n_i}$ for all $i = 1, ..., m$, stands for the block diagonal matrix, whose diagonal components are $f_1, ..., f_m$ matrices. The set $Co(x, y) = \{\lambda x + (1 - \lambda)y, 0 \leq \lambda \leq 1\}$ is the convex hull of $\{x, y\}$. The notation $e_n(i) \in \mathbb{R}^n$ refers to a column vector, whose elements are zero, except the i-th one:

$$e_n(i) = \underbrace{[0 \dots 0 \overbrace{\ 1\ }^{\text{i-th}} 0 \dots 0]^T}_{\text{n elements}}$$

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II. Preliminaries and Problem Statement

The classic discrete-time TS fuzzy model is a convex combination of linear models, having the form:

$$\begin{aligned} x(k+1) &= \sum_{l=1}^{s} h_l(z(k))(A_l x(k) + B_l u(k)) \\ y(k) &= \sum_{l=1}^{s} h_l(z(k)) C_l x(k), \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$ is the state vector, $u(k) \in \mathbb{R}^{n_u}$ is the control input, $y(k) \in \mathbb{R}^{n_y}$ is the measured output vector, s is the number of rules, $z(k) \in \mathbb{R}^{n_z}$ is the premise vector, and h_l, $l = 1, ..., s$ are nonlinear functions with the property

$$h_l \in [0,\, 1],\, l = 1, ..., s, \quad \sum_{l=1}^{s} h_l(z) = 1. \tag{2}$$

These nonlinear functions are called the membership functions. Matrices A_l, B_l, and C_l represent the l-th local model. Throughout this paper, the following shorthand notations are used to represent convex sums of matrix expressions:

$$F_z = \sum_{l=1}^{s} h_l(z(k)) F_l, \quad F_{z+} = \sum_{l=1}^{s} h_l(z(k+1)) F_l. \tag{3}$$

Based on this notation, (1) can be rewritten as

$$\begin{aligned} x(k+1) =& A_z x(k) + B_z u(k) \\ y(k) =& C_z x(k). \end{aligned} \tag{4}$$

A. Lemmas and properties

In order to develop our results we will use the following properties and lemmas.

Property 1 ([16]): Let T and R be matrices of appropriate dimensions and ranks, with $R = R^T > 0$. Then
$-T^T R^{-1} T \leq -T - T^T + R$.

Property 2 ([16]): (Schur complement). Let $\mathcal{M} = \mathcal{M}^T = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}$, with M_{11} and M_{22} square matrices of appropriate dimensions. Then:

$$\mathcal{M} < 0 \Leftrightarrow \begin{cases} M_{22} < 0 \\ M_{11} - M_{12} M_{22}^{-1} M_{12}^T < 0 \end{cases} \tag{5}$$

Lemma 1 ([16]): (Congruence) Given matrix $P = P^T$ and a full column rank matrix Q, it holds that
$P > 0 \quad \Rightarrow \quad QPQ^T > 0.$

Estimation and control problems are often defined as triple-sum negativity problems having the form

$$F_{zzz+} = \sum_{l_1=1}^{s} \sum_{l_2=1}^{s} \sum_{l_3=1}^{s} h_{l_1}(z) h_{l_2}(z) h_{l_3}(z+) F_{l_1 l_2 l_3} < 0, \tag{6}$$

with symmetric matrices $F_{l_1 l_2 l_3}$ and nonlinear functions h_l satisfying the convex sum property in (2).

Lemma 2 ([17]): Equation (6) is satisfied if the following conditions hold

$$\begin{aligned} F_{l_1 l_1 l_3} <& 0 \\ \frac{2}{s-1} F_{l_1 l_1 l_3} + F_{l_1 l_2 l_3} + F_{l_2 l_1 l_3} <& 0, \end{aligned}$$

for all $l_1, l_2, l_3 = 1, ..., s$, $l_1 \neq l_2$.

A useful inequality was presented in [8], which provides a powerful condition for the LMI problem.

Lemma 3 ([8]): Let X and Y be two given matrices of appropriate dimensions. Then, for any symmetric positive definite matrix S of appropriate dimension, the following inequality holds:

$$X^T Y + Y^T X \leq \frac{1}{2} \big(X + SY\big)^T S^{-1} \big(X + SY\big). \tag{7}$$

We also use a slightly different form of (7):

$$X^T Y + Y^T X \leq \frac{1}{2} \big(S^{-1} X + Y\big)^T S \big(S^{-1} X + Y\big). \tag{8}$$

For globally Lipschitz functions we use the following lemma.

Lemma 4 ([10]): Let $\phi : \mathbb{R}^n \to \mathbb{R}^q$ be a differentiable function on $\mathbb{R}^n$. Then the following items are equivalent:

- ϕ is globally Lipschitz
- there exist finite scalar constants a_{ij} and b_{ij}, so that for all $v, r \in \mathbb{R}^n$ there exist $\zeta_i \in Co(v, r)$, $\zeta_i \neq v$, $\zeta_i \neq r$ and functions $\psi_{ij} : \mathbb{R}^n \to \mathbb{R}$ satisfying the following:

$$\begin{aligned} \phi(v) - \phi(r) &= \sum_{i=1}^{q} \sum_{j=1}^{n} \psi_{ij}(\zeta_i) \mathcal{H}_{ij}(v - r) \\ a_{ij} &\leq \psi_{ij}(\zeta_i) \leq b_{ij} \\ \psi_{ij} &= \frac{\partial \phi_i}{\partial v_j}(\zeta_i), \quad \mathcal{H}_{ij} = e_q(i)\, e_n(j)^T, \end{aligned}$$

where v_j refers to the j-th element in the v vector, and

$$e_n(i) = \underbrace{[0 \ldots 0 \overbrace{\;1\;}^{\text{i−th}} 0 \ldots 0]^T}_{\text{n elements}}, \;\; e_q(j) = \underbrace{[0 \ldots 0 \overbrace{\;1\;}^{\text{j−th}} 0 \ldots 0]^T}_{\text{q elements}}.$$

B. Problem statement

The model considered has the following structure:

$$\begin{aligned} x(k+1) =& A_z x(k) + G\gamma(x(k)) + g(y(k), u(k)) + Ew(k) \\ y(k) =& C_z x(k) + Dw(k), \end{aligned} \tag{9}$$

where A_z and C_z have the same meaning as in (1). The disturbance term is denoted with $w(k) \in \mathbb{R}^{n_w}$, with E and D the corresponding matrices. The vector $g(y(k), u(k)) \in \mathbb{R}^{n_x}$ contains the terms that depend on the input and the output.

It is assumed that the premise vector $z(k)$ depends only on measured states and inputs. The nonlinear terms that depend on unmeasured states are handled by the nonlinear consequents. The quantity $\gamma(x(k)) \in \mathbb{R}^m$ contains the nonlinear consequents. It is an m-dimensional column vector, and the following can be written:

$$G\gamma(x(k)) = \sum_{i=1}^{m} G_i \gamma_i(\overbrace{H_i x(k)}^{\nu_i}), \tag{10}$$

where $\gamma_i(\cdot)$ is the i-th nonlinear function from the $\gamma(\cdot)$ vector, and G_i denotes the i-th column of G. Moreover, $H_i \in \mathbb{R}^{n_i \times n_x}$, where n_i denotes the number of inputs of the i-th nonlinearity, and $\nu_i = H_i x(k)$. The observer we propose has the following structure:

$$\begin{aligned}\hat{x}(k+1) =& A_z\hat{x}(k) + \sum_{i=1}^{m} G_i\gamma_i(\hat{\nu}_i) + g(y(k),u(k)) \\ &+ Q_z^{-1}L_z(y(k) - C_z\hat{x}(k)),\\ \hat{\nu}_i =& H_i\hat{x}(k) + K_{iz}\big(y(k) - C_z\hat{x}(k)\big),\ \ L_z = \sum_{l=1}^{s} h_l(z(k))L_l,\\ Q_z =& \sum_{l=1}^{s} h_l(z(k))Q_l,\ \ K_{iz} = \sum_{l=1}^{s} h_l(z(k))K_{il}.\end{aligned} \tag{11}$$

An extra degree of freedom is added via the term Q_z. It is assumed that $\gamma(\cdot)$ is globally Lipschitz. Based on Lemma 4 the following can be written:

$$\begin{aligned}&G\big(\gamma(x(k)) - \gamma(\hat{x}(k))\big) = \sum_{i=1}^{m}\sum_{j=1}^{n_i} \Phi_{ij}(\zeta_i)\mathcal{H}_{ij}(\nu_i - \hat{\nu}_i)\\ &\mathcal{H}_{ij} = G_i e_{n_i}(j)^T, \quad \Phi_{ij}(\zeta_i) = \frac{\partial \gamma_i}{\partial \nu_{ij}}(\zeta_i)\\ &a_{ij} \le \Phi_{ij}(\zeta_i) \le b_{ij},\end{aligned} \tag{12}$$

where ν_{ij} denotes the j-th element from the vector ν_i. Note that in (12) the G matrix is included in $\mathcal{H}_{ij}$, so a simpler form is obtained. The model defined in (9) can be reformulated so that $a_{ij} = 0$, for all $i = 1, ..., m$, $j = 1, ..., n_i$. For more details on these modifications we refer the reader to [3], [7]. In what follows we denote: $\Phi_{ij} := \Phi_{ij}(\zeta_i)$ and we consider $a_{ij} = 0$, for all $i = 1, ..., m$, $j = 1, ..., n_i$. We denote the error with $e(k) := x(k) - \hat{x}(k)$, and based on (9)-(11) we have the following error dynamics:

$$\begin{aligned}e(k+1) =& A_z x(k) + G\gamma(x(k)) + g(y(k),u(k)) + Ew(k)\\ &- A_z\hat{x}(k) - G\gamma(\hat{x}(k)) - g(y(k),u(k))\\ &- Q_z^{-1}L_z(y(k) - C_z\hat{x}(k))\\ =& (A_z - Q_z^{-1}L_zC_z)e(k) + G\big(\gamma(x(k)) - \gamma(\hat{x}(k))\big)\\ &+ (E - Q_z^{-1}L_zD)w(k).\end{aligned} \tag{13}$$

Using (12), we can rewrite

$$\begin{aligned}G\big(\gamma(x(k)) - \gamma(\hat{x}(k))\big) &= \sum_{i=1}^{m}\sum_{j=1}^{n_i}\Phi_{ij}\mathcal{H}_{ij}(\nu_i - \hat{\nu}_i)\\ &= \sum_{i=1}^{m}\sum_{j=1}^{n_i}\Phi_{ij}\mathcal{H}_{ij}\Big((H_i - K_{iz}C_z)e(k) - K_{iz}Dw(k)\Big).\end{aligned}$$

This leads to

$$\begin{aligned}e(k+1) =& \left(\mathbb{A}_L + \sum_{i=1}^{m}\sum_{j=1}^{n_i}\Phi_{ij}\mathcal{H}_{ij}\mathbb{H}_{K_i}\right)e(k)\\ &+ \left(\mathbb{E}_L + \sum_{i=1}^{m}\sum_{j=1}^{n_i}\Phi_{ij}\mathcal{H}_{ij}\mathbb{D}_{K_i}\right)w(k),\end{aligned} \tag{14}$$

where

$$\begin{aligned}\mathbb{A}_L =& A_z - Q_z^{-1}L_zC_z, \quad \mathbb{H}_{K_i} = H_i - K_{iz}C_z,\\ \mathbb{E}_L =& E - Q_z^{-1}L_zD, \quad \mathbb{D}_{K_i} = K_{iz}D.\end{aligned} \tag{15}$$

We consider the following fuzzy Lyapunov function

$$V(e(k)) := e(k)^T P_z e(k), \tag{16}$$

for which the difference across time steps is:

$$\Delta V := e(k+1)^T P_{z+} e(k+1) - e(k)^T P_z e(k). \tag{17}$$

We add an H_∞ performance signal: $c(k) = Je(k)$, $\mu > 0$, and formulate the following criterion:

$$\mathcal{W}_k := \Delta V + \|c(k)\|^2 - \mu\|w(k)\|^2 \le 0. \tag{18}$$

Now we are ready to present the main results of this paper.

III. Main Results

Theorem 1 presents the improvements we provide on Theorem 2 of [10]. Afterwards, Corollaries 1 and 2 give sufficient LMI conditions to satisfy Theorem 1 following the lines of conditions in [10], while Theorem 2 provides another approach to obtain sufficient LMI conditions.

Theorem 1: Consider the error dynamics in (14), and the H_∞ performance index formulated in (18). If there exist matrices $P_{l_1} = P_{l_1}^T > 0$, Q_{l_2}, $\mathbb{S}_{ij} = \mathbb{S}_{ij}^T > 0$, and K_{il_2}, L_{l_2}, for $i = 1, ..., m$, $j = 1, ..., n_i$ and $l_1, l_2, l_3 = 1, ..., s$, such that Lemma 2 holds with

$$F_{l_1l_2l_3} = \begin{bmatrix} M_{l_1l_2l_3} & [\Omega_{1l_2} \dots \Omega_{ml_2}] \\ * & -\Lambda\mathbb{S}\end{bmatrix} \tag{19}$$

where

$$M_{l_1l_2l_3} = \begin{bmatrix} -P_{l_1} + J^TJ & 0 & A_{l_1}^TQ_{l_2}^T - C_{l_1}^TL_{l_2}^T \\ * & -\mu I & E^TQ_{l_2}^T - D^TL_{l_2}^T \\ * & * & P_{l_3} - Q_{l_2} - Q_{l_2}^T\end{bmatrix}$$

$$\Omega_{il_2} = \left[\Pi_{i1l_2} \dots \Pi_{in_il_2}\right], \ \Pi_{ijl_2} = \begin{bmatrix}(H_i^T - C_{l_1}^TK_{il_2}^T)\mathbb{S}_{ij} \\ (D^TK_{il_2}^T)\mathbb{S}_{ij} \\ Q_{l_2}\mathcal{H}_{ij}\end{bmatrix}$$

$$\begin{aligned}\Lambda =& \text{bdiag}(\Lambda_1, ..., \Lambda_m),\ \Lambda_i = \text{bdiag}\left(\frac{2}{b_{i1}}I, ..., \frac{2}{b_{in_i}}I\right)\\ \mathbb{S} =& \text{bdiag}\big(\mathbb{S}_1, ..., \mathbb{S}_m\big),\ \mathbb{S}_i = \text{bdiag}\big(\mathbb{S}_{i1}, ..., \mathbb{S}_{in_i}\big),\end{aligned}$$

then the H_∞ performance condition defined in (18) is satisfied.

Proof: By calculating $\mathcal{W}_k$ along the trajectories of $e(k)$, we obtain:

$$\begin{aligned}\mathcal{W}_k =& e(k)^T\left[\big(\mathbb{A}_L + \mathbb{M}\big)^T P_{z+}\big(\mathbb{A}_L + \mathbb{M}\big) - P_z + J^TJ\right]e(k)\\ &+ w(k)^T\left[\big(\mathbb{E}_L + \mathbb{N}\big)^T P_{z+}\big(\mathbb{A}_L + \mathbb{N}\big) - \mu I\right]w(k)\\ &+ 2e(k)^T\left[\big(\mathbb{A}_L + \mathbb{M}\big)^T P_{z+}\big(\mathbb{E}_L + \mathbb{N}\big)\right]w(k)\end{aligned} \tag{20}$$

where

$$\mathbb{M} = \sum_{i=1}^{m}\sum_{j=1}^{n_i}\Phi_{ij}\mathcal{H}_{ij}\mathbb{H}_{K_i},\ \ \mathbb{N} = \sum_{i=1}^{m}\sum_{j=1}^{n_i}\Phi_{ij}\mathcal{H}_{ij}\mathbb{D}_{K_i}. \tag{21}$$

This can be written as

$$\mathcal{W}_k = \begin{bmatrix} e(k) \\ w(k)\end{bmatrix}^T \Sigma \begin{bmatrix} e(k) \\ w(k)\end{bmatrix},$$

where

$$\Sigma = \begin{bmatrix} -P_z + J^T J & 0 \\ 0 & -\mu I \end{bmatrix} + \begin{bmatrix} (\mathbb{A}_L + \mathbb{M})^T \\ (\mathbb{E}_L + \mathbb{N})^T \end{bmatrix} P_{z+} \begin{bmatrix} \mathbb{A}_L + \mathbb{M} & \mathbb{E}_L + \mathbb{N} \end{bmatrix} \tag{22}$$

Since $\Sigma < 0$ implies $\mathcal{W}_k < 0$, in what follows we consider only Σ. By applying the Schur complement on (22) and congruence with $\mathrm{bdiag}[I\ I\ Q_z]$ the following condition is obtained:

$$\begin{bmatrix} -P_z + J^T J & 0 & (\mathbb{A}_L + \mathbb{M})^T Q_z^T \\ * & -\mu I & (\mathbb{E}_L + \mathbb{N})^T Q_z^T \\ * & * & -Q_z P_{z+}^{-1} Q_z^T \end{bmatrix} < 0. \tag{23}$$

Using Property 1 on $Q_z P_{z+} Q_z^T$, we have:

$$\begin{bmatrix} -P_z + J^T J & 0 & (\mathbb{A}_L + \mathbb{M})^T Q_z^T \\ * & -\mu I & (\mathbb{E}_L + \mathbb{N})^T Q_z^T \\ * & * & P_{z+} - Q_z - Q_z^T \end{bmatrix} < 0. \tag{24}$$

Next, by following the steps in the proof of [10] we can separate (24) as:

$$\begin{bmatrix} -P_z + J^T J & 0 & \mathbb{A}_L^T Q_z^T \\ * & -\mu I & \mathbb{E}_L^T Q_z^T \\ * & * & P_{z+} - Q_z - Q_z^T \end{bmatrix} + \sum_{i=1}^{m} \sum_{j=1}^{n_i} \Phi_{ij} \left(\begin{bmatrix} 0 \\ 0 \\ Q_z \mathcal{H}_{ij} \end{bmatrix} \begin{bmatrix} \mathbb{H}_{K_i} & \mathbb{D}_{K_i} & 0 \end{bmatrix} + * \right) < 0 \tag{25}$$

We denote

$$\mathbb{X}_{ij} = \begin{bmatrix} 0 & 0 & \mathcal{H}_{ij}^T Q_z^T \end{bmatrix}, \quad \mathbb{Y}_i = \begin{bmatrix} \mathbb{H}_{K_i} & \mathbb{D}_{K_i} & 0 \end{bmatrix}, \tag{26}$$

based on Lemma 4 the following inequality holds:

$$\mathbb{X}_{ij}^T \mathbb{Y}_i + \mathbb{X}_{ij} \mathbb{Y}_i^T \le \frac{1}{2} (\mathbb{X}_{ij} + \mathbb{S}_{ij} \mathbb{Y}_i)^T \mathbb{S}_{ij}^{-1} (\mathbb{X}_{ij} + \mathbb{S}_{ij} \mathbb{Y}_i) \tag{27}$$

Since $0 \le \Phi_{ij} \le b_{ij}$, see (12), (25) holds if:

$$\begin{bmatrix} -P_z + J^T J & 0 & \mathbb{A}_L^T Q_z^T \\ * & -\mu I & \mathbb{E}_L^T Q_z^T \\ * & * & P_{z+} - Q_z - Q_z^T \end{bmatrix} + \sum_{i=1}^{m} \sum_{j=1}^{n_i} (\mathbb{X}_{ij} + \mathbb{S}_{ij} \mathbb{Y}_i)^T \left(\frac{2}{b_{ij}} \mathbb{S}_{ij} \right)^{-1} (\mathbb{X}_{ij} + \mathbb{S}_{ij} \mathbb{Y}_i) < 0 \tag{28}$$

Next, by applying the Schur complement on (28) we get:

$$\begin{bmatrix} M_{zzz+} & [\Omega_{1z} \dots \Omega_{mz}] \\ * & -\Lambda \mathbb{S} \end{bmatrix} < 0$$
$$M_{zzz+} = \begin{bmatrix} -P_z + J^T J & 0 & A_z^T Q_z^T - C_z^T L_z^T \\ * & -\mu I & E^T Q_z^T - D^T L_z^T \\ * & * & P_{z+} - Q_z - Q_z^T \end{bmatrix} \tag{29}$$

By applying Lemma 2 we obtain (19). ∎

To highlight the advantages of the presented approach, first we focus on $\mathbb{X}_{ij}$ in (26). In [10] the term $\mathbb{X}_{ij}$ depends on $\mathcal{H}_{ij}^T P^T$, where P must be a positive definite symmetric matrix, and $\mathcal{H}_{ij}$ is a $n_x \times n_i$ matrix, in which the j-th column is G_i and the rest of the columns are 0. Since P has to be symmetric, the $\mathbb{X}_{ij}$ term makes the design conditions restrictive.

This restrictive form is relaxed in our design. We use a non-symmetric fuzzy matrix $Q_z = \sum_{l=1}^{s} h_l(z(k)) Q_l$, which introduces unconstrained decision variables in $\mathbb{X}_{ij}$. Matrix Q_z also helps in M_{zzz+} in (29), since the observer is completely decoupled from the Lyapunov function, while in Theorem 2 in [10] the observer gains are computed as $L_z = P^{-1} \mathcal{X}_z$, where P is from the Lyapunov function.

On the other hand, in the Lyapunov synthesis, instead of the quadratic Lyapunov function $e(k)^T P e(k)$ we use a fuzzy Lyapunov function, having the form: $e^T(k) P_z e(k)$. This makes the design conditions less restrictive, see [18]. In Theorem 1, if we take a quadratic Lyapunov function, so $P_l = P$ for all $l = 1, ..., s$, and constant Q, with $Q = P$, then we obtain Theorem 2 of [10]. We can conclude that the design condition presented in Theorem 2 of [10] is a special case of Theorem 1.

Although the conditions defined in (19) are less restrictive than those in Theorem 2 in [10], they are still bilinear due to the terms $K_{il_2} \mathbb{S}_{ij}$ for all $i = 1, ..., m$, $j = 1, ..., n_i$. The change of variables, like $\mathcal{X} = K_{il_2} \mathbb{S}_{ij}$, is not possible since $\mathbb{S}_{ij}$ depends on both i and j. Two solutions are proposed in [10] to obtain sufficient LMI conditions in form of corollaries. The first corollary imposes $\mathbb{S}_{ij} = \mathcal{S}_i$, so that a variable change can be used: $\mathcal{Y}_i = K_i^T \mathcal{S}_i$, while the second corollary considers $K_i = 0$. In what follows we also formulate two corollaries for the options mentioned above.

Corollary 1: Consider the error dynamics in (14), and the H_∞ performance index formulated in (18). If there exist matrices $P_{l_1} = P_{l_1}^T > 0$, Q_{l_2}, $\mathcal{S}_i = \mathcal{S}_i^T > 0$, and $\mathcal{Y}_{il_2}$, L_{l_2}, for $i = 1, ..., m$ and $l_1, l_2, l_3 = 1, ..., s$, such that Lemma 2 holds with (19), where

$$\mathbb{S}_i = \mathrm{bdiag}(\underbrace{\mathcal{S}_i, ..., \mathcal{S}_i}_{n_i})$$

then the H_∞ performance condition defined in (18) is satisfied, and the observer gains can be recovered from $K_{il_1} = \mathcal{S}_i^{-1} \mathcal{Y}_{il_1}^T$.

Proof: The proof is the same as that of Theorem 1 with $\mathbb{S}_{ij} = \mathcal{S}_i$. ∎

Corollary 2: Consider the error dynamics in (14), and the H_∞ performance index formulated in (18). If there exist matrices $P_{l_1} = P_{l_1}^T > 0$, Q_{l_2}, $\mathbb{S}_{ij} = \mathbb{S}_{ij}^T > 0$, for $i = 1, ..., m$, $j = 1, ..., n_i$ and $l_1, l_2, l_3 = 1, ..., s$, such that Lemma 2 holds with (19), then the H_∞ performance condition defined in (18) is satisfied.

Proof: The proof is as above with $K_{il_2} = 0$. ∎

Next, we provide a novel approach as an alternative to obtain sufficient LMI conditions.

Theorem 2: Consider the error dynamics in (14), and the H_∞ performance index formulated in (18). If there exist matrices $P_{l_1} = P_{l_1}^T > 0$, Q_{l_2}, $\mathbb{S}_{ij} = \mathbb{S}_{ij}^T > 0$, where the j-th column of $\mathbb{S}_{ij}$ is $\underbrace{[0 \dots 0 \overbrace{\alpha_{ij}}^{\mathrm{j-th}} 0 \dots 0]^T}_{\mathrm{n_i\ elements}}$, with constants

α_{ij}, and K_{il_2}, L_{l_2}, for $i = 1, ..., m$, $j = 1, ..., n_i$ and $l_1, l_2, l_3 = 1, ..., s$, such that Lemma 2 holds with

$$F_{l_1 l_2 l_3} = \begin{bmatrix} M_{l_1 l_2 l_3} & [\Omega_{1l_2} \dots \Omega_{ml_2}] \\ * & -\Lambda \mathbb{S} \end{bmatrix} \tag{30}$$

where

$$M_{l_1 l_2 l_3} = \begin{bmatrix} -P_{l_1} + J^T J & 0 & A_{l_1}^T Q_{l_2}^T - C_{l_1}^T L_{l_2}^T \\ * & -\mu I & E^T Q_{l_2}^T - D^T L_{l_2}^T \\ * & * & P_{l_3} - Q_{l_2} - Q_{l_2}^T \end{bmatrix}$$

$$\Omega_{il_2} = \begin{bmatrix} \Pi_{i1} & \dots & \Pi_{in_i} \end{bmatrix}, \quad \Pi_{ij} = \begin{bmatrix} H_i^T - C_{l_1}^T K_{il_2}^T \\ D^T K_{il_2}^T \\ \alpha_{ij} Q_{l_2} \mathcal{H}_{ij} \end{bmatrix}$$

$$\Lambda = \mathrm{bdiag}(\Lambda_1, ..., \Lambda_m), \ \Lambda_i = \mathrm{bdiag}\left(\frac{2}{b_{i1}} I, ..., \frac{2}{b_{in_i}} I\right)$$
$$\mathbb{S} = \mathrm{bdiag}(\mathbb{S}_1, ..., \mathbb{S}_m), \ \mathbb{S}_i = \mathrm{bdiag}(\mathbb{S}_{i1}, ..., \mathbb{S}_{in_i}), \tag{31}$$

then the H_∞ performance condition in (18) is satisfied.

Proof: We follow the line of the proof of Theorem 1, equations (20)-(26). After that, we consider inequality (8), and the following is obtained:

$$\mathbb{X}_{ij}^T \mathbb{Y}_i + \mathbb{X}_{ij} \mathbb{Y}_i^T \le \frac{1}{2} \left(\tilde{\mathbb{S}}_{ij}^{-1} \mathbb{X}_{ij} + \mathbb{Y}_i\right)^T \tilde{\mathbb{S}}_{ij} \left(\tilde{\mathbb{S}}_{ij}^{-1} \mathbb{X}_{ij} + \mathbb{Y}_i\right), \tag{32}$$

which leads to

$$\begin{bmatrix} -P_z + J^T J & 0 & \mathbb{A}_L^T Q_z^T \\ * & -\mu I & \mathbb{E}_L^T Q_z^T \\ * & * & P_{z+} - Q_z - Q_z^T \end{bmatrix} + \sum_{i=1}^{m} \sum_{j=1}^{n_i} \left(\tilde{\mathbb{S}}_{ij}^{-1} \mathbb{X}_{ij} + \mathbb{Y}_i\right)^T \left(\frac{2}{b_{ij}} \tilde{\mathbb{S}}_{ij}^{-1}\right)^{-1} \left(\tilde{\mathbb{S}}_{ij}^{-1} \mathbb{X}_{ij} + \mathbb{Y}_i\right) < 0 \tag{33}$$

We denote $\mathbb{S}_{ij} := \tilde{\mathbb{S}}_{ij}^{-1}$. Now we examine

$$\mathbb{S}_{ij} \mathbb{X}_{ij} + \mathbb{Y}_i = \begin{bmatrix} \mathbb{H}_{K_i} & \mathbb{D}_{K_i} & \mathbb{S}_{ij} \mathcal{H}_{ij}^T Q_z^T \end{bmatrix} \tag{34}$$

The bilinear term is $\mathbb{S}_{ij} \mathcal{H}_{ij}^T Q_z^T$, which can be written as

$$\mathbb{S}_{ij} \mathcal{H}_{ij}^T Q_z^T = \mathbb{S}_{ij} e_{n_i}(j) G_i^T Q_z^T, \tag{35}$$

where the multiplication of $\mathbb{S}_{ij} e_{n_i}(j)$ results in the j-th column of $\mathbb{S}_{ij}$. The j-th column is $\alpha_{ij} e_{n_i}(j)$, and we have:

$$\begin{bmatrix} -P_z + J^T J & 0 & \mathbb{A}_L^T Q_z^T \\ * & -\mu I & \mathbb{E}_L^T Q_z^T \\ * & * & P_{z+} - Q_z - Q_z^T \end{bmatrix} + \sum_{i=1}^{m} \sum_{j=1}^{n_i} \begin{bmatrix} \mathbb{H}_{K_i}^T \\ \mathbb{D}_{K_i}^T \\ \alpha_{ij} Q_z \mathcal{H}_{ij} \end{bmatrix} \left(\frac{2\mathbb{S}_{ij}}{b_{ij}}\right)^{-1} \begin{bmatrix} \mathbb{H}_{K_i} & \mathbb{D}_{K_i} & \mathcal{H}_{ij}^T Q_z^T \alpha_{ij} \end{bmatrix} < 0 \tag{36}$$

Using the Schur complement, we obtain:

$$\begin{bmatrix} M_{zzz+} & [\Omega_{1z} \dots \Omega_{mz}] \\ * & -\Lambda \mathbb{S} \end{bmatrix} < 0$$

$$\Omega_{iz} = \begin{bmatrix} \Pi_{i1z} & \dots & \Pi_{in_i z} \end{bmatrix}, \ \Pi_{ijz} = \begin{bmatrix} H_i^T - C_z^T K_{iz}^T \\ D^T K_{iz}^T \\ \alpha_{ij} Q_z \mathcal{H}_{ij} \end{bmatrix}, \tag{37}$$

and the rest of the elements are as in Theorem 1. Sufficient LMI conditions are obtained applying Lemma 2. ■

The main advantage of Theorem 2 is that even if it imposes a restrictive structure, still every $\mathcal{H}_{ij}$ term has a corresponding $\mathbb{S}_{ij}$ matrix, and not a single $\mathcal{S}_i$ for all the j-s like in Corollary 1. If we have a good initial guess for the parameters α_{ij}, the problem becomes a simple LMI. Alternatively, we can search for α_{ij} in a logarithmically spaced family of values $\alpha_{ij} \in \{10^{-6}, 10^{-5}, ..., 10^5, 10^6\}$. On the other hand, if we have many α_{ij} parameters the problem can become computationally intractable.

IV. Comparison with the State of the Art

In order to highlight the advantages obtained by our approach, we compare Corollaries 1 and 2 from [10] with our approaches presented in Corollaries 1 and 2, and Theorem 2 on a numerical example.

Example 1: Consider model (9) with two local models and the following matrices:

$$A_1 = \begin{bmatrix} 1 & 0.008 \\ 0 & 0.29 \end{bmatrix}, \ A_2 = \begin{bmatrix} 1 & 0.008 \\ 0 & 0.96 \end{bmatrix}, \ G = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \ H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ E = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \ D = 0.02, \tag{38}$$

Moreover, $g(y(k), u(k)) = 0$ and we consider one nonlinearity denoted with γ and it fulfills (12) with $a_{11} = a_{12} = 0$, $b_{12} = 0.5$. This example is simple enough so standard LMI solvers can easily handle, but has a nonlinearity that depends on two states, so approaches in [19], [3], [4] are not suitable for it.

TABLE I
Minimum value of μ for Example 1

b_{11}	Cor. 1 [10]	Cor. 2 [10]	Cor. 1	Cor. 2	Th. 2
2	1.0203	0.8754	0.8263	0.6544	0.5642
2.2	1.0581	0.9022	0.8517	0.6738	0.5644
2.4	1.0954	0.9287	0.8768	0.6931	0.5646
2.6	1.1324	0.9551	0.9019	0.7122	0.5648
2.8	1.1690	0.9814	0.9270	0.7314	0.5650
3	1.2053	1.0077	0.9520	0.7505	0.5652

We study the minimum μ what we can obtain based on b_{11}. To solve the LMI problem, the `sedumi` solver is used in the `Yalmip` [20] framework. To fit in the framework of the linear parameter varying models used in [10], we consider the parameter $\rho \in [0.03, 0.7]$, and matrices
$$A_0 = \begin{bmatrix} 1 & 0.008 \\ 0 & 0.99 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The results obtained are presented in Table I. It can be seen that Corollaries 1-2 outperform the corollaries presented in [10] in terms of the minimum value of μ. The best results are obtained with Theorem 2, but with the added computational complexity of finding a suitable α_{11} and α_{12}. Regarding the computational complexity of solving the LMI conditions, according to [16], page 18, a realistic approximation of the numerical complexity using the interior-point method used by `sedumi` is $\mathcal{O}(N_d^{2.1} N_l^{1.2})$, where N_d is the number of scalar decision variables and N_l is the row size of the LMI problem. For the conditions in Corollary 1 we have $N_d = 25$, for Corollary 2 this value is $N_d = 24$, and for Theorem 2 we have $N_d = 24$, and for all the cases $N_l = 56$. To show the advantage of using the fuzzy Q_z matrix we provide another example dedicated only to this purpose.

Example 2: We use (9) with the following matrices:

$$C_1 = C_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; C_2 = C_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \omega_1 & 0 \end{bmatrix}$$

$$A_1 = A_2 = \begin{bmatrix} 1 & 0.001 & 0 & 0 \\ 0.1 & 0.2 & 0.1 & 0.3 \\ 0 & 0 & 1 & 0.001 \\ 0.1 & -0.3 & 0.2 & \omega_2 \end{bmatrix}, \; G = \begin{bmatrix} 1 & -2 \\ -4 & 7 \\ 3 & -1 \\ -12 & 9 \end{bmatrix}$$

$$A_3 = A_4 = \begin{bmatrix} 1 & 0.001 & 0 & 0 \\ 0.1 & 0.2 & 0.1 & 0.3 \\ 0 & 0 & 1 & 0.001 \\ 0.1 & -0.3 & 0.2 & 1.1 \end{bmatrix}, \tag{39}$$

where ω_1 and ω_2 are two parameters. We consider the following two cases:

1.) Corollary 1 with $P_z = P$, $Q_z = P$, equivalent to [10].
2.) Corollary 1 with $P_z = P$.

We vary the values of ω_1 and ω_2 in the range $\omega_1 = [0.8,\ 2.2]$ and $\omega_2 = [-0.2,\ 1.2]$, and we look for feasible solutions with respect to the matrices in (39). The results obtained can be seen on Fig. 1. We can see that a much wider range of systems can be handled by adding the extra degree of freedom with Q_z.

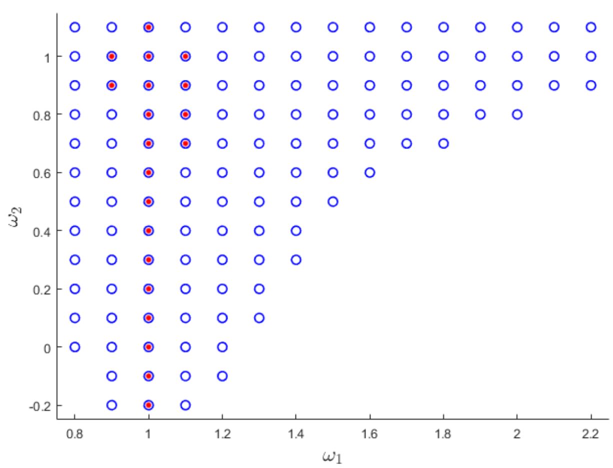

Fig. 1. '.'-Corollary 1 with $P_z = P$, $Q_z = P$, 'o'-Corollary 1 with $P_z = P$.

V. Conclusions and Future Work

This paper presented a novel approach to observer design for discrete-time nonlinear systems with nonlinear consequents. The design exploits the TS fuzzy framework together with globally Lipschitz nonlinearities. An H_∞ performance index was used, and the conditions were formulated so that the effect of the disturbance was minimised. To highlight the novelty of the paper two examples were presented.

There are many future directions, among which we will focus on extending this work to observer-based controller design. On the other hand we plan to consider a wider range of models, for example by considering fuzzy terms also for the G matrix.

References

[1] R. E. Kalman, "On the general theory of control systems," *IRE Transaction on Automatic Control, 4, pp. 481-492*, 1959.

[2] D. G. Luenberger, "An introduction to observers," *Transaction on Automatic Control, 16, pp. 596-602*, 1971.

[3] M. Arcak and P. Kokotović, "Observer-based control of systems with slope-restricted nonlinearities," *Transaction on Automatic Control, 46, pp. 1146-1150*, 2001.

[4] M. Chong, R. Postoyan, D. Nesic, L. Kuhlmann, and A. Varsavsky, "A robust circle criterion observer with application to neural mass models," *Automatica, Elsevier, 48 (11), pp.2986-2989*, 2012.

[5] A. Zemouche, M. Boutayeb, and G. I. Bara, "Observer design for nonlinear systems: An approach based on the differential mean value theorem." in *Proceedings of the 44th IEEE Conference on Decision and Control*. IEEE, 2005, pp. 6353–6358.

[6] ——, "Observers for a class of Lipschitz systems with extension to H_∞ performance analysis," *Systems & Control Letters*, vol. 57, no. 1, pp. 18–27, 2008.

[7] A. Zemouche and M. Boutayeb, "On LMI conditions to design observers for Lipschitz nonlinear systems," *Automatica*, vol. 49, no. 2, pp. 585–591, 2013.

[8] A. Zemouche, R. Rajamani, B. Boulkroune, H. Rafaralahy, and M. Zasadzinski, "H_∞ circle criterion observer design for Lipschitz nonlinear systems with enhanced LMI conditions," in *2016 American Control Conference (ACC)*. IEEE, 2016, pp. 131–136.

[9] K. C. Draa, H. Voos, M. Alma, A. Zemouche, and M. Darouach, "LMI-based trajectory tracking for a class of nonlinear systems with application to an anaerobic digestion process," *Annual American Control Conference, Wisconsin Center, Milwaukee, USA, pp. 4393-4397*, 2018.

[10] K. Chaib Draa, A. Zemouche, M. Alma, H. Voos, and M. Darouach, "A discrete-time nonlinear state observer for the anaerobic digestion process," *International Journal of Robust and Nonlinear Control*, vol. 29, no. 5, pp. 1279–1301, 2019.

[11] J. Dong, Y. Wang, and G.-H. Yang, "Output feedback fuzzy controller design with local nonlinear feedback laws for discrete-time nonlinear systems," *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, vol. 40, no. 6, pp. 1447–1459, 2010.

[12] Z. Nagy and Zs. Lendek, "Observer-based controller design for Takagi-Sugeno fuzzy systems with local nonlinearities," *International Conference on Fuzzy Systems FuzzIEEE, New Orleans, LA, USA*, 2019.

[13] ——, "Observer Design for Discrete-Time TS Fuzzy Systems with Local Nonlinearities," in *Proceedings of the European Control Conference, Saint Petersburg, Russia*, 2020, pp. 1–6.

[14] H. Moodi and M. Farrokhi, "Robust observer design for Sugeno systems with incremental quadratic nonlinearity in the consequent," *International Journal of Applied Mathematics and Computer Science, 23, pp. 711-723*, 2013.

[15] ——, "Robust observer-based controller design for Takagi Sugeno systems with nonlinear consequent parts," *Fuzzy Sets and Systems, 273, pp. 141-154*, 2015.

[16] S. Boyd, L. El Ghaoui, E. Féron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, ser. Studies in Applied Mathematics. Philadelphia, PA, USA: Society for Industrial and Applied Mathematics, 1994.

[17] H. Tuan, P. Apkarian, T. Narikiyo, and Y. Yamamoto, "Parameterized linear matrix inequality techniques in fuzzy control system design," *IEEE Transactions on Fuzzy Systems*, vol. 9, no. 2, pp. 324–332, 2001.

[18] T.-M. Guerra, H. Kerkeni, J. Lauber, and L. Vermeiren, "An efficient Lyapunov function for discrete T–S models: observer design," *IEEE Transactions on Fuzzy Systems*, vol. 20, no. 1, pp. 187–192, 2011.

[19] M. Arcak and P. Kokotović, "Nonlinear observers: A circle criterion design," *Conference on Decision and Control, Phoenix, AZ, USA, pp. 4872-4876*, 1999.

[20] J. Löfberg, "YALMIP : A Toolbox for Modeling and Optimization in MATLAB," in *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004, pp. 284–289.